

More on the Terwilliger algebra of Johnson schemes

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Abstract

In [F. Levstein, C. Maldonado, The Terwilliger algebra of the Johnson schemes, Discrete Math. 307 (2007) 1621–1635], the Terwilliger algebra of the Johnson scheme $J(n, d)$ was determined when $n \geq 3d$. In this paper, we determine the Terwilliger algebra \mathcal{T} for the remaining case $2d \leq n < 3d$.

Key words: Association scheme; Johnson scheme; Terwilliger algebra.

1 Introduction

Let $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ denote a commutative association scheme, where X is a finite set. Suppose $\text{Mat}_X(\mathbb{C})$ denotes the algebra over \mathbb{C} consisting of all matrices whose rows and columns are indexed by X . For each i , let A_i denote the binary matrix in $\text{Mat}_X(\mathbb{C})$ whose (x, y) -entry is 1 if and only if $(x, y) \in R_i$. We call A_i the *i th adjacency matrix* of \mathcal{X} . We abbreviate $A = A_1$, and call it the *adjacency matrix* of \mathcal{X} . The subalgebra of $\text{Mat}_X(\mathbb{C})$ spanned by A_0, A_1, \dots, A_d is called the *Bose-Mesner algebra* of \mathcal{X} , denoted by \mathcal{B} . Since \mathcal{B} is commutative and generated by real symmetric matrices, it has a basis consisting of primitive idempotents, denoted by $E_0 = \frac{1}{|X|}J, E_1, E_2, \dots, E_d$. For each $i \in \{0, 1, \dots, d\}$, write

$$A_i = \sum_{j=0}^d p_i(j)E_j, \quad E_i = \frac{1}{|X|} \sum_{j=0}^d q_i(j)A_j.$$

The scalars $p_i(j)$ and $q_i(j)$ are called the *eigenvalues* and the *dual eigenvalues* of \mathcal{X} , respectively.

Fix $x \in X$. For $0 \leq i \leq d$, let E_i^* denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ whose (y, y) -entry is defined by

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } (x, y) \in R_i, \\ 0, & \text{otherwise.} \end{cases}$$

The subalgebra \mathcal{T} of $\text{Mat}_X(\mathbb{C})$ generated by $A_0, A_1, \dots, A_d; E_0^*, E_1^*, \dots, E_d^*$ is called the *Terwilliger algebra* of \mathcal{X} with respect to x .

Terwilliger [12] first introduced the Terwilliger algebra of association schemes, which is an important tool in considering the structure of an association scheme. For more information; see [4, 5, 13, 14]. The Terwilliger algebra is a finite-dimensional semisimple \mathbb{C} -algebra, it is difficult to determine its structure in general. The structures of the Terwilliger algebras of some association schemes were determined; see [1, 2] for group schemes, [15] for strongly

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regular graphs, [7, 10] for Hamming schemes, [11] for Johnson schemes, [8] for odd graphs, [9] for incidence graphs of Johnson geometry.

Let $[n]$ denote the set $\{1, 2, \dots, n\}$ and $\binom{[n]}{d}$ denote the collection of all d -element subsets of $[n]$. For $0 \leq i \leq d$, define $R_i = \{(x, y) \in \binom{[n]}{d} \times \binom{[n]}{d} \mid |x \cap y| = d - i\}$. Then $(\binom{[n]}{d}, \{R_i\}_{i=0}^d)$ is a symmetric association scheme of class d , which is called the *Johnson scheme*, denoted by $J(n, d)$. Note that $J(n, d)$ is isomorphic to $J(n, n - d)$. So we always assume that $n \geq 2d$.

In [11], the Terwilliger algebra of the Johnson scheme $J(n, d)$ was determined when $n \geq 3d$. In this paper, we focus on the remaining case, and determine the Terwilliger algebra \mathcal{T} of $J(n, d)$. In Section 2, we introduce intersection matrices and some useful identities. In Sections 3, two families of subalgebras $\mathcal{M}^{(n, d)}$ and \mathcal{N} of $\text{Mat}_X(\mathbb{C})$ are constructed. In the last two sections, we show that $\mathcal{T} = \mathcal{M}^{(n, d)}$ and $\mathcal{T} = \mathcal{N}$ according to $n > 2d$ and $n = 2d$, respectively.

2 Intersection matrix

In this section we first introduce some useful identities for intersection matrices, then describe the adjacency matrix of the Johnson scheme $J(n, d)$ in terms of intersection matrices.

Let $H_{i,j}^r(v)$ be a binary matrix whose rows and columns are indexed by the elements of $\binom{[v]}{i}$ and $\binom{[v]}{j}$ respectively, whose $\alpha_i \alpha_j$ -entry is 1 if and only if $|\alpha_i \cap \alpha_j| = r$. For simplicity, write $H_{i,j} := H_{i,j}^{\min(i,j)}$. Now we introduce some useful identities for intersection matrices.

Lemma 2.1 ([6, Theorem 3]) *For $0 \leq l \leq \min(i, j)$ and $0 \leq s \leq \min(j, k)$,*

$$H_{i,j}^l(v) H_{j,k}^s(v) = \sum_{g=0}^{\min(i,k)} \left(\sum_{h=0}^g \binom{g}{h} \binom{i-g}{l-h} \binom{k-g}{s-h} \binom{v+g-i-k}{j+h-l-s} \right) H_{i,k}^g(v).$$

Lemma 2.2 ([11, Lemma 4.4]) *Let v be a positive integer.*

(i) *For $0 \leq i \leq j \leq l \leq v$,*

$$H_{i,j}(v) H_{j,l}(v) = \binom{l-i}{l-j} H_{i,l}.$$

(ii) *For $0 \leq \max(i, l) \leq j \leq v$,*

$$H_{i,j}(v) H_{j,l}(v) = \sum_{m=0}^{j-\max(i,l)} \binom{v-\max(i,l)-m}{j-\max(i,l)-m} H_{i,l}^{\min(i,l-m)}(v).$$

(iii) *For $0 \leq j \leq \min(i, l) \leq v$,*

$$H_{i,j}(v) H_{j,l}(v) = \sum_{m=0}^{\min(i,l)-j} \binom{\min(i,l)-m}{j} H_{i,l}^{\min(i,l-m)}(v).$$

Pick $x \in \binom{[n]}{d}$. For $0 \leq i \leq d$, write $\Omega_i := \{y \in \binom{[n]}{d} \mid |x \cap y| = d - i\}$. Then we have the partition $\binom{[n]}{d} = \dot{\cup}_{i=0}^d \Omega_i$. Now we consider the m th adjacent matrix A_m of $J(n, d)$ as a block matrix with respect to this partition. Denote $(A_m)_{|\Omega_i \times \Omega_j|}$ the submatrix of A_m with rows indexed by Ω_i and columns indexed by Ω_j .

In the remaining of this paper, we always assume that $I^{(v,k)}$ denotes the identity matrix of size $\binom{v}{k}$ and $A_m^{(v,k)}$ denotes the m -th adjacency matrix of $J(v, k)$. In fact $A_m^{(v,k)} = H_{k,k}^m(v)$.

Lemma 2.3 ([11]) *For $0 \leq i < j \leq d$, we have*

$$\begin{aligned} A_{|\Omega_i \times \Omega_i} &= I^{(d,d-i)} \otimes A^{(n-d,i)} + A^{(d,d-i)} \otimes I^{(n-d,i)}, \\ A_{|\Omega_i \times \Omega_{i+1}} &= H_{d-i,d-i-1}(d) \otimes H_{i,i+1}(n-d), \\ A_{|\Omega_i \times \Omega_j} &= 0, \quad \text{if } j \geq i+2, \end{aligned}$$

where “ \otimes ” denotes the Kronecker product of matrices.

3 Two algebras

Let $2d \leq n$ and $X = \binom{[n]}{d}$. In this section we shall construct two subalgebras $\mathcal{M}^{(n,d)}$ and \mathcal{N} of $\text{Mat}_X(\mathbb{C})$, which are in fact the Terwilliger algebras \mathcal{T} of $J(n,d)$ according to $2d < n < 3d$ and $J(2d,d)$.

Let $\mathcal{B}^{(v,k)}$ denote the Bose-Mesner algebra of $J(v,k)$, and $\{E_r^{(v,k)}\}_{r=0}^{\min(k,v-k)}$ denote its primitive idempotents. Let $p_i^{(v,k)}(j)$ be the eigenvalue of $A_i^{(v,k)}$ satisfying $A_i^{(v,k)} E_j^{(v,k)} = p_i^{(v,k)}(j) E_j^{(v,k)}$.

For $i, j = 0, 1, \dots, d$, define the vector space

$$M_{i,j}^{(n,d)} = (\mathcal{B}^{(d,d-i)} H_{d-i,d-j}(d)) \otimes (\mathcal{B}^{(n-d,i)} H_{i,j}(n-d)).$$

Let $L : Y \mapsto L(Y)$ be a linear map from $M_{i,j}^{(n,d)}$ to $\text{Mat}_X(\mathbb{C})$, where $(L(Y))_{\Omega_l \times \Omega_m}$ is Y if $l = i$ and $m = j$; 0 otherwise. Define

$$\mathcal{M}^{(n,d)} = \bigoplus_{i,j=0}^d L(M_{i,j}^{(n,d)}). \quad (1)$$

Lemma 3.1 ([11, Lemma 4.4]) *Let $R(v,k,h) := \{r \mid H_{k,h}^r(v) \neq 0\}$. Then $\mathcal{B}^{(v,k)} H_{k,h}(v) = H_{k,h}(v) \mathcal{B}^{(v,h)} = \langle \{H_{k,h}^r(v)\}_{r \in R(v,k,h)} \rangle$.*

Lemma 3.1 implies that $H_{d-i,d-j}^r(d) \otimes H_{i,j}^s(n-d)$, $r \in R(d,d-i,d-j)$, $s \in R(n-d,i,j)$ is a basis of $M_{i,j}^{(n,d)}$. By Lemma 2.1, we observe that $\mathcal{M}^{(n,d)}$ is an algebra.

By Lemmas 2.3 and 3.1, the adjacency matrix A of $J(n,d)$ belongs to $\mathcal{M}^{(n,d)}$. Since each m th adjacency matrix of $J(n,d)$ may be written as a polynomial of A , one gets $\mathcal{B}^{(n,d)} \subseteq \mathcal{M}^{(n,d)}$. The fact that $E_m^* = L(H_{d-m,d-m}(d) \otimes H_{m,m}(n-d)) \in \mathcal{M}^{(n,d)}$ implies that \mathcal{T} is a subalgebra of $\mathcal{M}^{(n,d)}$.

Next we construct another algebra \mathcal{N} . For $i, j = 0, 1, \dots, d$, let $N_{i,j}$ be the vector space generated by

$$(E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)})(H_{d-i,d-j}(d) \otimes H_{i,j}(d)), \quad 0 \leq r \leq s \leq \min(d, d-i). \quad (2)$$

Define

$$\mathcal{N} = \bigoplus_{i,j=0}^d L(N_{i,j}). \quad (3)$$

Observe that $\mathcal{N} \subseteq \mathcal{M}^{(2d,d)}$.

Lemma 3.2 *Each vector space $N_{i,j}$ has the basis*

$$H_{d-i,d-j}^{d-i-j+g}(d) \otimes H_{i,j}^h(d) + H_{d-i,d-j}^{d-i-j+h}(d) \otimes H_{i,j}^g(d), \quad g, h \in R(d,i,j), \quad g \leq h. \quad (4)$$

Proof. By Lemma 3.1 we have that $\mathcal{B}^{(v,k)} H_{k,h}(v) = H_{k,h}(v) \mathcal{B}^{(v,h)} = \langle \{H_{k,h}^r(v)\}_{r \in R(v,k,h)} \rangle$. Also note that it holds

$$H_{d-i,d-j}^{d-i-j+g}(d) \otimes H_{i,j}^l(d) + H_{d-i,d-j}^{d-i-j+l}(d) \otimes H_{i,j}^g(d) = H_{i,j}^g(d) \otimes H_{i,j}^l(d) + H_{i,j}^l(d) \otimes H_{i,j}^g(d),$$

and that $\{H_{i,j}^g(d) \otimes H_{i,j}^h(d)\}_{g,h \in R(i,j)}$ are linearly independent, which implies that $\{H_{i,j}^g(d) \otimes H_{i,j}^h(d) + H_{i,j}^h(d) \otimes H_{i,j}^g(d)\}_{g,h \in R(d,i,j), g \leq h}$ are linearly independent. Hence, the desired result follows. \square

Theorem 3.3 *Let \mathcal{N} be as in (3). Then \mathcal{N} is an algebra.*

Proof. It suffices to show that $N_{i,j}N_{j,k} \subseteq N_{i,k}$. By Lemma 2.1 there exist scalars $\alpha_{l,s}^g$ such that

$$H_{i,j}^l(d)H_{j,k}^s(d) = \sum_{g=0}^{\min(i,k)} \alpha_{l,s}^g H_{i,k}^g(d),$$

which implies that

$$\begin{aligned} & (H_{i,j}^m(d) \otimes H_{i,j}^n(d) + H_{i,j}^n(d) \otimes H_{i,j}^m(d))(H_{j,k}^s(d) \otimes H_{j,k}^t(d) + H_{j,k}^t(d) \otimes H_{j,k}^s(d)) \\ &= \sum_{g=0}^{\min(i,k)} \sum_{l=0}^{\min(i,k)} (\alpha_{m,s}^g \alpha_{n,t}^l + \alpha_{m,t}^g \alpha_{n,s}^l) (H_{i,k}^g(d) \otimes H_{i,k}^l(d) + H_{i,k}^l(d) \otimes H_{i,k}^g(d)), \end{aligned}$$

as desired. \square

Lemma 3.4 *Let \mathcal{T} be the Terwilliger algebra of $J(2d, d)$. Then \mathcal{T} is a subalgebra of \mathcal{N} .*

Proof. Observe that $A_{|\Omega_i \times \Omega_{i+l}} = 0$ for $l \geq 2$. By Lemma 2.3, we obtain $A_{|\Omega_i \times \Omega_{i+1}} = H_{d-i,d-i-1}(d) \otimes H_{i,i+1}(d)$, and

$$\begin{aligned} A_{|\Omega_i \times \Omega_i} &= I^{(d,d-i)} \otimes A^{(d,i)} + A^{(d,d-i)} \otimes I^{(d,i)} \\ &= \sum_{r=0}^{\min(i,d-i)} \sum_{s=0}^{\min(i,d-i)} (p_1^{(d,i)}(r) + p_1^{(d,i)}(s)) (E_r^{(d,d-i)} \otimes E_s^{(d,i)}) \\ &= \sum_{r=0}^{\min(i,d-i)} \sum_{s=0}^{\min(i,d-i)} \frac{p_1^{(d,i)}(r) + p_1^{(d,i)}(s)}{2} (E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)}). \end{aligned}$$

Then $A \in \mathcal{N}$. Also we have that $E_m^* = L(H_{d-m,d-m}(d) \otimes H_{m,m}(d)) \in \mathcal{N}$, so $\mathcal{T} \subseteq \mathcal{N}$. \square

We next introduce two mappings proposed in [11]: the lift map denoted by \mathcal{L}_i and the pullback map denoted by \mathcal{P}_i .

For $0 \leq i < d$, define \mathcal{L}_i be the linear mapping from $M_{i,i}^{(n,d)}$ to $M_{i+1,i+1}^{(n,d)}$ satisfying

$$\begin{aligned} & \mathcal{L}_i(E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}) \\ &= (H_{d-i-1,d-i}(d)E_r^{(d,d-i)}H_{d-i,d-i-1}(d)) \otimes (H_{i+1,i}(n-d)E_s^{(n-d,i)}H_{i,i+1}(n-d)); \end{aligned}$$

for $0 < i \leq d$, define \mathcal{P}_i be the linear mapping from $M_{i,i}^{(n,d)}$ to $M_{i-1,i-1}^{(n,d)}$ satisfying

$$\begin{aligned} & \mathcal{P}_i(E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}) \\ &= (H_{d-i+1,d-i}(d)E_r^{(d,d-i)}H_{d-i,d-i+1}(d)) \otimes (H_{i-1,i}(n-d)E_s^{(n-d,i)}H_{i,i-1}(n-d)). \end{aligned}$$

Note that the lift map is defined premultiplying by $(H_{d-i-1,d-i}(d) \otimes H_{i+1,i}(n-d))$ (that is equal to $A_{|\Omega_{i+1} \times \Omega_i}$ by Lemma 2.3) and post multiplying by $(H_{d-i,d-i-1}(d) \otimes H_{i,i+1}(n-d))$ (that is equal to $A_{|\Omega_i \times \Omega_{i+1}}$ by Lemma 2.3). Since they belong to $\mathcal{T}_{|\Omega_{i+1} \times \Omega_i}$ and $\mathcal{T}_{|\Omega_i \times \Omega_{i+1}}$ respectively and since \mathcal{T} is an algebra, then $\mathcal{L}_i(Y) \in \mathcal{T}_{|\Omega_{i+1} \times \Omega_{i+1}}$ for any $Y \in \mathcal{T}_{|\Omega_i \times \Omega_i}$. Similarly $\mathcal{P}_i(Y) \in \mathcal{T}_{|\Omega_{i-1} \times \Omega_{i-1}}$ for any $Y \in \mathcal{T}_{|\Omega_i \times \Omega_i}$.

By [3, p.220], for $0 \leq j \leq k$,

$$p_1^{(v,k)}(j) = (k-j)(v-k-j) - j, \quad p_k^{(v,k)}(j) = (-1)^j \binom{v-k-j}{k-j}. \quad (5)$$

Write $l_{v,k,j} = v - k + p_1^{(v,k)}(j)$ and $p_{v,k,j} = k + p_1^{(v,k)}(j)$.

Lemma 3.5 ([11, Lemma 5.6])

$$\begin{aligned} \mathcal{L}_i(E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}) &= p_{d,d-i,r} l_{n-d,i,s} E_r^{(d,d-i-1)} \otimes E_s^{(n-d,i+1)}, \\ \mathcal{P}_i(E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}) &= l_{d,d-i,r} p_{n-d,i,s} E_r^{(d,d-i+1)} \otimes E_s^{(n-d,i-1)}. \end{aligned}$$

Here, $E_j^{(v,k)} = 0$ if $j > \min(k, v-k)$.

Corollary 3.6 [11, Corollary 5.7]

$$(E_r^{(d,d-i)} H_{d-i,d-j}(d)) \otimes (E_s^{(n-d,i)} H_{i,j}(n-d)) = (H_{d-i,d-j}(d) E_r^{(d,d-j)}) \otimes (H_{i,j}(n-d) E_s^{(n-d,j)}).$$

4 \mathcal{T} -algebra of $J(n, d)$ for $2d < n < 3d$

In this section we always assume that $2d < n < 3d$ and $\mathcal{M}^{(n,d)}$ is as in (1). We shall prove that $\mathcal{M}^{(n,d)}$ is the Terwilliger algebra \mathcal{T} of $J(n, d)$.

For any real number a , we have

$$(A + aI)_{|\Omega_i \times \Omega_i} = \sum_{r=0}^{\min(i,d-i)} \sum_{s=0}^{\min(i,n-d-i)} (\mu_{i,r} + \lambda_{i,s} + a) E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}, \quad (6)$$

where $\mu_{i,r} = p_1^{(d,d-i)}(r)$ and $\lambda_{i,s} = p_1^{(n-d,i)}(s)$. We always assume that a is a real number large enough such that the coefficients in (6) are positive.

Remark 1 If these coefficients are pairwise distinct, each $E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}$ belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$ since the left hand side of (6) and its powers belong to $\mathcal{T}_{|\Omega_i \times \Omega_i}$. By orthogonality of the idempotents

$$(A + aI)_{|\Omega_i \times \Omega_i}^j = \sum_{r=0}^{\min(i,d-i)} \sum_{s=0}^{\min(i,n-d-i)} (\mu_{i,r} + \lambda_{i,s} + a)^j E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}, \quad (7)$$

obtaining a linear system of equations given by the powers of $(A + aI)_{|\Omega_i \times \Omega_i}$ as linear combinations of $E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}$ with a Vandermonde matrix. See Section 5.1 in [11] for more explanation.

Theorem 4.1 Suppose $2d < n < 3d$. Let \mathcal{T} be the Terwilliger algebra of $J(n, d)$ and $\mathcal{M}^{(n,d)}$ be the algebra as in (1). Then $\mathcal{T} = \mathcal{M}^{(n,d)}$.

Proof. Since $L(M_{i,j}^{(n,d)}) = L(M_{i,i}^{(n,d)}(H_{d-i,d-j}(d) \otimes H_{i,j}(n-d)))$, by [11, Proposition 5.2] it is sufficient to prove that, for $0 \leq r \leq \min(d-i, i)$ and $0 \leq s \leq \min(n-d-i, i)$,

$$E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \in \mathcal{T}_{|\Omega_i \times \Omega_i}. \quad (8)$$

We shall prove (8) by induction on i (i decreases from d to 0).

Comment 1 Since it holds $\mathcal{P}_{i+1}(Y) \in \mathcal{T}_{|\Omega_i \times \Omega_i}$ for any $Y \in \mathcal{T}_{|\Omega_{i+1} \times \Omega_{i+1}}$ the strategy of the proof is to pull it back those projectors $E_r^{(d,d-i-1)} \otimes E_s^{(n-d,i+1)} \in \mathcal{T}_{|\Omega_i \times \Omega_{i+1}}$ (whenever its pullback is different from zero) or separate the projectors as we explained in the Remark 1.

Induction: observe that $(A + aI)_{|\Omega_d \times \Omega_d} = \sum_{s=0}^d (\mu_{d,0} + \lambda_{d,s} + a) E_0^{(d,0)} \otimes E_s^{(n-d,d)}$. Since $\lambda_{d,s}$'s are pairwise distinct, (8) holds for $i = d$.

Case 1. $\lceil \frac{n-d}{2} \rceil + 1 \leq i \leq d-1$.

Note that in these case $\min(d-i, i) = d-i$ and $\min(n-d-i, i) = n-d-i$.

For $0 \leq r \leq d-i-1$ and $0 \leq s \leq n-d-i-1$, by (5) one gets $l_{d,d-i-1,r} \neq 0$ and $p_{n-d,i+1,s} \neq 0$. By Lemma 3.5,

$$E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} = l_{d,d-i-1,r}^{-1} p_{n-d,i+1,s}^{-1} \mathcal{P}_{i+1}(E_r^{(d,d-i-1)} \otimes E_s^{(n-d,i+1)});$$

and so $E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \in \mathcal{T}_{|\Omega_i \times \Omega_i}$.

It is not possible pull back $(E_{d-i}^{(d,d-i-1)} \otimes E_s^{(n-d,i+1)})$ not also $(E_r^{(d,d-i-1)} \otimes E_{n-d-i}^{(n-d,i+1)})$.

By (6), we have

$$\begin{aligned} (A + aI)_{|\Omega_i \times \Omega_i} &= \sum_{r=0}^{d-i-1} \sum_{s=0}^{n-d-i-1} (\mu_{i,r} + \lambda_{i,s} + a) E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \\ &= \sum_{s=0}^{n-d-i} (\mu_{i,d-i} + \lambda_{i,s} + a) E_{d-i}^{(d,d-i)} \otimes E_s^{(n-d,i)} + \sum_{r=0}^{d-i-1} (\mu_{i,r} + \lambda_{i,n-d-i} + a) E_r^{(d,d-i)} \otimes E_{n-d-i}^{(n-d,i)}. \end{aligned}$$

It follows that the right hand side of the equality belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$. In order to show that (8) holds, it suffices to show that each term belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$. Observe that there do not exist three coefficients with the same value. If there exists a term whose coefficient is different from other coefficients, then this term belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$. Next suppose that there exist two terms with the same coefficient. Suppose that $\mu_{i,d-i} + \lambda_{i,q} + a = \mu_{i,u} + \lambda_{i,n-d-i} + a$. Then $E_{d-i}^{(d,d-i)} \otimes E_q^{(n-d,i)} + E_u^{(d,d-i)} \otimes E_{n-d-i}^{(n-d,i)}$ belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$, and by Lemma 3.5 its image under \mathcal{P}_i is

$$(i + \mu_{i,d-i})(i + \lambda_{i,q}) E_{d-i}^{(d,d-i+1)} \otimes E_q^{(n-d,i-1)} + (i + \mu_{i,u})(i + \lambda_{i,n-d-i}) E_u^{(d,d-i+1)} \otimes E_{n-d-i}^{(n-d,i-1)}.$$

Suppose $(i + \mu_{i,d-i})(i + \lambda_{i,q}) = (i + \mu_{i,u})(i + \lambda_{i,n-d-i})$. Since $\mu_{i,d-i} + \lambda_{i,q} = \mu_{i,u} + \lambda_{i,n-d-i}$, one gets $\mu_{i,d-i} \lambda_{i,q} = \mu_{i,u} \lambda_{i,n-d-i}$. It follows that $(\mu_{i,d-i} - \lambda_{i,n-d-i})(\mu_{i,d-i} - \mu_{i,u}) = 0$, a contradiction to $\mu_{i,d-i} \neq \lambda_{i,n-d-i}$ and $\mu_{i,d-i} \neq \mu_{i,u}$. Therefore, we have $(i + \mu_{i,d-i})(i + \lambda_{i,q}) \neq (i + \mu_{i,u})(i + \lambda_{i,n-d-i})$, which implies that both $E_{d-i}^{(d,d-i+1)} \otimes E_q^{(n-d,i-1)}$ and $E_u^{(d,d-i+1)} \otimes E_{n-d-i}^{(n-d,i-1)}$ belong to $\mathcal{T}_{|\Omega_{i-1} \times \Omega_{i-1}}$. Computing their image under \mathcal{L}_{i-1} , by Lemma 3.5 again $E_{d-i}^{(d,d-i)} \otimes E_q^{(n-d,i)}$ and $E_u^{(d,d-i)} \otimes E_{n-d-i}^{(n-d,i)}$ belong to $\mathcal{T}_{|\Omega_i \times \Omega_i}$, as desired.

Case 2. $i = \lceil \frac{n-d}{2} \rceil$.

We divide our discussion into two subcases.

Case 2.1. $n-d$ is odd. By Lemma 3.5, for any $E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \in M_{i,i}^{(n,d)}$,

$$\mathcal{P}_i(E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}) = l_{d,d-i,r} p_{n-d,i,s} E_r^{(d,d-i+1)} \otimes E_s^{(n-d,i-1)} \neq 0.$$

Similar to the proof in Case 1, (8) holds.

Case 2.2. $n - d$ is even. For $0 \leq r \leq d - i - 1$ and $0 \leq s \leq n - d - i - 1$, by Lemma 3.5,

$$E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} = l_{d,d-i-1,r}^{-1} p_{n-d,i+1,s}^{-1} \mathcal{P}_{i+1}(E_r^{(d,d-i-1)} \otimes E_s^{(n-d,i+1)}),$$

which implies that $E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \in \mathcal{T}_{|\Omega_i \times \Omega_i}$.

Next we consider $r = d - i$ or $s = n - d - i$. Write $\mu'_{i,r} = p_{d-i}^{(d,d-i)}(r)$ and $\lambda'_{i,s} = p_i^{(n-d,i)}(s)$. Since $(Ad)_{|\Omega_i \times \Omega_i} = A_{d-i}^{(d,d-i)} \otimes A_i^{(n-d,i)}$,

$$\begin{aligned} & (Ad + aI)_{|\Omega_i \times \Omega_i} - \sum_{r=0}^{d-i-1} \sum_{s=0}^{n-d-i-1} (\mu'_{i,r} \lambda'_{i,s} + a) E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \\ &= \sum_{s=0}^{n-d-i} (\mu'_{i,d-i} \lambda'_{i,s} + a) E_{d-i}^{(d,d-i)} \otimes E_s^{(n-d,i)} + \sum_{r=0}^{d-i-1} (\mu'_{i,r} \lambda'_{i,n-d-i} + a) E_r^{(d,d-i)} \otimes E_{n-d-i}^{(n-d,i)}. \end{aligned}$$

It follows that the right hand side of the equality belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$. By (5), observe that $\mu'_{i,r} \lambda'_{i,n-d-i} + a$ is not equal to any other coefficient. Then $E_r^{(d,d-i)} \otimes E_{n-d-i}^{(n-d,i)} \in \mathcal{T}_{|\Omega_i \times \Omega_i}$ for $0 \leq r \leq d - i - 1$.

By (6),

$$\sum_{s=0}^{n-d-i} (\mu_{i,d-i} + \lambda_{i,s} + a) E_{d-i}^{(d,d-i)} \otimes E_s^{(n-d,i)}$$

belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$. Moreover, its coefficients are pairwise distinct, so $E_{d-i}^{(d,d-i)} \otimes E_s^{(n-d,i)}$ belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$ for $0 \leq s \leq n - d - i$. Therefore, (8) holds.

Case 3. $\lceil \frac{d}{2} \rceil \leq i \leq \lceil \frac{n-d}{2} \rceil - 1$.

Note that in these case $\min(d - i, i) = d - i$ and $\min(n - d - i, i) = i$.

Similarly, by Lemma 3.5 we have that $E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}$ belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$ for $0 \leq r \leq d - i - 1$ and $0 \leq s \leq i$.

By (6) again, the matrix

$$\sum_{s=0}^i (\mu_{i,d-i} + \lambda_{i,s} + a) E_{d-i}^{(d,d-i)} \otimes E_s^{(n-d,i)}$$

belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$. Moreover, its coefficients are pairwise distinct, so (8) holds.

Case 4. $0 \leq i \leq \lceil \frac{d}{2} \rceil - 1$.

Note that in these case $\min(d - i, i) = i$ and $\min(n - d - i, i) = i$.

By Lemma 3.5 again, (8) holds. □

Next we shall decompose \mathcal{T} as a direct sum of some simple ideals.

For $0 \leq r \leq \lfloor \frac{d}{2} \rfloor$ and $0 \leq s \leq \lfloor \frac{n-d}{2} \rfloor$, define

$$\begin{aligned} e_{r,s} &= \min\{i \mid 0 \neq E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \in M_{i,i}^{(n,d)}\}, \\ d_{r,s} &= |\{i \mid 0 \neq E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \in M_{i,i}^{(n,d)}\}| - 1. \end{aligned}$$

Note that $e_{r,s} = \max(r, s)$ and $e_{r,s} + d_{r,s} = \min(d - r, n - d - s)$.

For $0 \leq r \leq \min(d - i, i)$ and $0 \leq s \leq \min(n - d - i, i)$, define

$${}^{rs}T_{ij} = (E_r^{(d,d-i)} H_{d-i,d-j}(d)) \otimes (E_s^{(n-d,i)} H_{i,j}(n-d)), \quad (9)$$

$${}^{rs}\mathcal{T} = \langle \{L({}^{rs}T_{i,j})\}_{0 \leq i,j \leq d} \rangle. \quad (10)$$

Proposition 4.2 *Let ${}^{rs}\mathcal{T}$ be as in (10). Then ${}^{rs}\mathcal{T}$ is an ideal of \mathcal{T} .*

Proof. It suffices to show that $L({}^{rs}T_{ij})L({}^{pq}T_{lm}) \in {}^{rs}\mathcal{T}$ and $L({}^{pq}T_{lm})L({}^{rs}T_{ij}) \in {}^{rs}\mathcal{T}$.

If $j \neq l$, then $L({}^{rs}T_{ij})L({}^{pq}T_{lm}) = 0$. Suppose $j = l$. Since $H_{d-i,d-j}(d) \otimes H_{i,j}(n-d) \in M_{i,j}^{(n,d)}$ and $H_{d-j,d-m}(d) \otimes H_{j,m}(n-d) \in M_{j,m}^{(n,d)}$, we obtain $(H_{d-i,d-j}(d)H_{d-j,d-m}(d)) \otimes (H_{i,j}(n-d)H_{j,m}(n-d)) \in M_{i,m}^{(n,d)}$. It follows that there exist scalars $\beta_{u,v}$ such that

$$\begin{aligned} & (H_{d-i,d-j}(d)H_{d-j,d-m}(d)) \otimes (H_{i,j}(n-d)H_{j,m}(n-d)) \\ &= \sum_{u=0}^{\min(i,d-i)} \sum_{v=0}^{\min(i,n-d-i)} \beta_{u,v} (E_u^{(d,d-i)} H_{d-i,d-m}(d)) \otimes (E_v^{(n-d,i)} H_{i,m}(n-d)). \end{aligned}$$

By Lemma 3.6,

$$\begin{aligned} & {}^{rs}T_{ij} {}^{pq}T_{jm} \\ &= \delta_{r,p} \delta_{s,q} (H_{d-i,d-j}(d) E_r^{(d,d-j)} H_{d-j,d-m}(d)) \otimes (H_{i,j}(n-d) E_s^{(d,j)} H_{j,m}(n-d)) \\ &= \delta_{r,p} \delta_{s,q} (E_r^{(d,d-i)} \otimes E_s^{(d,i)}) ((H_{d-i,d-j}(d) H_{d-j,d-m}(d)) \otimes (H_{i,j}(n-d) H_{j,m}(n-d))) \\ &= \delta_{r,p} \delta_{s,q} \beta_{r,s} (E_r^{(d,d-i)} H_{d-i,d-m}(d)) \otimes (E_s^{(d,i)} H_{i,m}(n-d)), \end{aligned}$$

where $\delta_{r,p}$ is the Kronecker delta; and so

$${}^{rs}T_{ij} {}^{pq}T_{jm} = \delta_{r,p} \delta_{s,q} \beta_{r,s} {}^{rs}T_{im}. \quad (11)$$

It follows that $L({}^{rs}T_{ij})L({}^{pq}T_{jm}) \in {}^{rs}\mathcal{T}$. Similarly $L({}^{pq}T_{lm})L({}^{rs}T_{ij}) \in {}^{rs}\mathcal{T}$. \square

By (11) we observe that ${}^{rs}\mathcal{T} {}^{pq}\mathcal{T} = \{0\}$ if and only if $(r, s) \neq (p, q)$. From the construction of $\mathcal{M}^{(n,d)}$, we have

$$\mathcal{T} = \bigoplus_{r=0}^{\lfloor d/2 \rfloor} \bigoplus_{s=0}^{\lfloor (n-d)/2 \rfloor} {}^{rs}\mathcal{T}.$$

Lemma 4.3 *Let ${}^{rs}T_{ij}$ be as in (9). Then ${}^{rs}T_{ij} \neq 0$ if and only if $i, j \in \{e_{r,s}, e_{r,s} + 1, \dots, e_{r,s} + d_{r,s}\}$.*

Proof. Note that $i, j \in \{\max(r, s), \dots, \min(d-r, n-d-s)\}$ if and only if $0 \leq r \leq \min(i, j, d-i, d-j)$ and $0 \leq s \leq \min(i, j, n-d-i, n-d-j)$. If r or s does not belong to above ranges, then ${}^{rs}T_{ij} = 0$ by Corollary 3.6. Since

$$H_{d-i,d-j}^r(d) \otimes H_{i,j}^s(n-d), \quad r \in R(d, d-i, d-j), s \in R(n-d, i, j)$$

is a basis of $M_{i,j}^{(n,d)}$, we have

$$\dim(M_{i,j}^{(n,d)}) = (\min(i, j, d-i, d-j) + 1) \times (\min(i, j, n-d-i, n-d-j) + 1).$$

All ${}^{rs}T_{ij}$'s generate $M_{i,j}^{(n,d)}$, so the desired result follows. \square

For $i, j \in \{e_{r,s}, e_{r,s} + 1, \dots, e_{r,s} + d_{r,s}\}$, write

$$n_{ij}^r = \begin{cases} \sum_{m=0}^{j-i} \binom{d-j-m}{d-i} p_m^{(d,d-i)}(r), & i \leq j, \\ \sum_{m=0}^{i-j} \binom{i-m}{j} p_m^{(d,d-i)}(r), & i \geq j; \end{cases}$$

$$n_s^{ij} = \begin{cases} \sum_{m=0}^{j-i} \binom{n-d-i-m}{j-i-m} p_m^{(n-d,i)}(s), & i \leq j, \\ \sum_{m=0}^{i-j} \binom{i-m}{j} p_m^{(n-d,i)}(s), & i \geq j. \end{cases}$$

By Lemma 2.2 we have $({}^{rs}T_{ij})({}^{rs}T_{ij})^T = n_{ij}^r n_s^{ij} E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \neq 0$. By computing the trace of this matrix, one gets $n_{ij}^r > 0$ and $n_s^{ij} > 0$. By (11), we may assume that ${}^{rs}T_{ij} {}^{rs}T_{jl} = \beta_{r,s}(i, j, l) {}^{rs}T_{il}$. Then $\beta_{r,s}(i, j, i) = n_{ij}^r n_s^{ij} > 0$. Taking the transpose on both sides of above equation, we obtain $\beta_{r,s}(i, j, l) = \beta_{r,s}(l, j, i)$. By Lemma 2.2 (i) and Lemma 3.6, we have $\beta_{r,s}(i, j, l) > 0$ if $i \geq j \geq l$. Note that

$$\begin{aligned} {}^{rs}T_{ij} {}^{rs}T_{jl} {}^{rs}T_{li} &= \beta_{r,s}(i, j, l) \beta_{r,s}(i, l, i) {}^{rs}T_{ii} = \beta_{r,s}(j, l, i) \beta_{r,s}(i, j, i) {}^{rs}T_{ii}. \\ {}^{rs}T_{li} {}^{rs}T_{ij} {}^{rs}T_{jl} &= \beta_{r,s}(l, i, j) \beta_{r,s}(l, j, l) {}^{rs}T_{ll} = \beta_{r,s}(i, j, l) \beta_{r,s}(l, i, l) {}^{rs}T_{ll}. \end{aligned}$$

Hence, we have $\beta_{r,s}(i, j, l) > 0$ for any $i, j, l \in \{\max(r, s), \dots, \min(d-r, n-d-s)\}$.

By Lemma 2.2 again,

$$({}^{rs}T_{ij} {}^{rs}T_{jl})({}^{rs}T_{ij} {}^{rs}T_{jl})^T = \frac{n_{ij}^r n_s^{ij} n_{jl}^r n_s^{jl}}{n_{il}^r n_s^{il}} {}^{rs}T_{il} ({}^{rs}T_{il})^T.$$

By (11), we have

$${}^{rs}T_{ij} {}^{rs}T_{jl} = \sqrt{\frac{n_{ij}^r n_s^{ij} n_{jl}^r n_s^{jl}}{n_{il}^r n_s^{il}}} {}^{rs}T_{il}. \quad (12)$$

Let $\text{Mat}_{d_{r,s}+1}(\mathbb{C})$ be the algebra consisting of all matrices with degree $d_{r,s} + 1$ whose rows and columns are indexed by $\{e_{r,s}, e_{r,s}+1, \dots, e_{r,s}+d_{r,s}\}$. Let $E_{i,j}$ be the matrix in $\text{Mat}_{d_{r,s}+1}(\mathbb{C})$ whose (i, j) -entry is 1 and others are 0.

Theorem 4.4 *Suppose $2d < n < 3d$. Let \mathcal{T} be the Terwilliger algebra of the Johnson scheme $J(n, d)$. Then*

$$\mathcal{T} \simeq \bigoplus_{r=0}^{\lfloor d/2 \rfloor} \bigoplus_{s=0}^{\lfloor (n-d)/2 \rfloor} \text{Mat}_{d_{r,s}+1}(\mathbb{C}).$$

Proof. It suffices to prove that ${}^{rs}\mathcal{T} \simeq \text{Mat}_{d_{r,s}+1}(\mathbb{C})$. Define the linear mapping ϕ from ${}^{rs}\mathcal{T}$ to $\text{Mat}_{d_{r,s}+1}(\mathbb{C})$ such that $\phi(L({}^{rs}T_{ij})) = \sqrt{n_{ij}^r n_s^{ij}} E_{i,j}$. By (12), we have ${}^{rs}\mathcal{T} \simeq \text{Mat}_{d_{r,s}+1}(\mathbb{C})$. \square

5 \mathcal{T} -algebra of $J(2d, d)$

Let \mathcal{N} be as in (3). In this section we shall prove that \mathcal{N} is the Terwilliger algebra \mathcal{T} of $J(2d, d)$.

Write $H_{d-i, d-j}^r := H_{d-i, d-j}^r(d)$ and $H_{i,j}^s := H_{i,j}^s(d)$ for simplicity.

Theorem 5.1 *Let \mathcal{T} be the Terwilliger algebra of $J(2d, d)$ and \mathcal{N} be the algebra as in (3). Then $\mathcal{T} = \mathcal{N}$.*

Proof. Since $L(N_{i,j}) = L(N_{i,i}(H_{d-i, d-j} \otimes H_{i,j}))$, by [11, Proposition 5.2] it is sufficient to prove that, for $0 \leq r \leq s \leq \min(d, d-i)$,

$$E_r^{(d, d-i)} \otimes E_s^{(d, i)} + E_s^{(d, d-i)} \otimes E_r^{(d, i)} \in \mathcal{T}_{|\Omega_i \times \Omega_i}. \quad (13)$$

We shall prove (13) by induction on i (i decreases from d to 0). For $i = d$, it is trivial.

Case 1. $\lceil \frac{d}{2} \rceil \leq i \leq d-1$.

For $0 \leq s \leq d-i-1$ and $0 \leq r \leq s$, by Lemma 3.5,

$$\begin{aligned} & E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)} \\ &= l_{d,d-i-1,r}^{-1} p_{d,i+1,s}^{-1} \mathcal{P}_{i+1}(E_r^{(d,d-i-1)} \otimes E_s^{(d,i+1)} + E_s^{(d,d-i-1)} \otimes E_r^{(d,i+1)}), \end{aligned}$$

which implies that $E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)} \in \mathcal{T}_{|\Omega_i \times \Omega_i}$. Write $\lambda_r = p_1^{(d,i)}(r)$. By Lemma 3.4, we have

$$\begin{aligned} & (A + aI)_{|\Omega_i \times \Omega_i} - \sum_{s=0}^{d-i-1} \sum_{r=0}^{d-i-1} \frac{\lambda_r + \lambda_s + a}{2} (E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)}) \\ &= \sum_{q=0}^{d-i-1} (\lambda_{d-i} + \lambda_q + a) (E_{d-i}^{(d,d-i)} \otimes E_q^{(d,i)} + E_q^{(d,d-i)} \otimes E_{d-i}^{(d,i)}) + (2\lambda_{d-i} + a) E_{d-i}^{(d,d-i)} \otimes E_{d-i}^{(d,i)}. \end{aligned}$$

Then the right hand side of the equality belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$. Moreover, its coefficients are pairwise distinct, so (13) holds.

Case 2. $0 \leq i \leq \lceil \frac{d}{2} \rceil - 1$.

By Lemma 3.5 again, (13) holds. \square

Next we shall decompose \mathcal{T} as a direct sum of some simple ideals.

For $0 \leq r, s \leq \frac{d}{2}$, define

$$\begin{aligned} e_{r,s} &= \min\{i \mid 0 \neq E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)} \in N_{i,i}\}, \\ d_{r,s} &= |\{i \mid 0 \neq E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)} \in N_{i,i}\}| - 1. \end{aligned}$$

Note that $e_{r,s} = \max(r, s)$ and $e_{r,s} + d_{r,s} = \min(d-r, d-s)$.

For $r, s \in \{0, 1, \dots, \min(d-i, i)\}$, define

$${}^{rs}T_{ij} = (E_r^{(d,d-i)} H_{d-i,d-j}) \otimes (E_s^{(d,i)} H_{i,j}) + (E_s^{(d,d-i)} H_{d-i,d-j}) \otimes (E_r^{(d,i)} H_{i,j}), \quad (14)$$

$${}^{rs}\mathcal{T} = \langle \{L({}^{rs}T_{ij})\}_{0 \leq i,j \leq d} \rangle. \quad (15)$$

Proposition 5.2 *Let ${}^{rs}\mathcal{T}$ be as in (15). Then ${}^{rs}\mathcal{T}$ is an ideal of \mathcal{T} .*

Proof. It suffices to show that $L({}^{rs}T_{ij})L({}^{pq}T_{lm}) \in {}^{rs}\mathcal{T}$ and $L({}^{pq}T_{lm})L({}^{rs}T_{ij}) \in {}^{rs}\mathcal{T}$.

If $j \neq l$, then $L({}^{rs}T_{ij})L({}^{pq}T_{lm}) = 0$. Suppose $j = l$. Since $H_{d-i,d-j} \otimes H_{i,j} \in N_{i,j}$ and $H_{d-j,d-m} \otimes H_{j,m} \in N_{j,m}$, we obtain $(H_{d-i,d-j} H_{d-j,d-m}) \otimes (H_{i,j} H_{j,m}) \in N_{i,m}$. It follows that there exist scalars $\beta_{u,v}$ such that

$$\begin{aligned} & (H_{d-i,d-j} H_{d-j,d-m}) \otimes (H_{i,j} H_{j,m}) \\ &= \sum_{v=0}^{\min(i,d-i)} \sum_{u=0}^v \beta_{u,v} (E_u^{(d,d-i)} \otimes E_v^{(d,i)} + E_v^{(d,d-i)} \otimes E_u^{(d,i)}) (H_{d-i,d-m} \otimes H_{i,m}). \end{aligned}$$

By Corollary 3.6,

$$\begin{aligned} & {}^{rs}T_{ij} {}^{pq}T_{jm} \\ &= \delta_{r,p} \delta_{s,q} (E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)}) ((H_{d-i,d-j} H_{d-j,d-m}) \otimes (H_{i,j} H_{j,m})) \\ &= \delta_{r,p} \delta_{s,q} \beta_{r,s} ((E_r^{(d,d-i)} H_{d-i,d-m}) \otimes (E_s^{(d,i)} H_{i,m}) + (E_s^{(d,d-i)} H_{d-i,d-m}) \otimes (E_r^{(d,i)} H_{i,m})), \end{aligned}$$

so we have

$${}^{rs}T_{ij} {}^{pq}T_{jm} = \delta_{r,p} \delta_{s,q} \beta_{r,s} {}^{rs}T_{im}. \quad (16)$$

It follows that $L({}^{rs}T_{ij})L({}^{pq}T_{jm}) \in {}^{rs}\mathcal{T}$. Similarly $L({}^{pq}T_{lm})L({}^{rs}T_{ij}) \in {}^{rs}\mathcal{T}$. \square

By (16) we observe that ${}^{rs}\mathcal{T}{}^{pq}\mathcal{T} = \{0\}$ if and only if $(r, s) \neq (p, q)$. From the construction of \mathcal{N} , we have

$$\mathcal{T} = \bigoplus_{s=0}^{\lfloor d/2 \rfloor} \bigoplus_{r=0}^s {}^{rs}\mathcal{T}.$$

Lemma 5.3 *Let ${}^{rs}T_{ij}$ be as in (14). Then ${}^{rs}T_{ij} \neq 0$ if and only if $i, j \in \{e_{r,s}, \dots, e_{r,s} + d_{r,s}\}$.*

Proof. The proof is similar to that of Lemma 4.3 and will be omitted. \square

For $i, j \in \{\max(r, s), \dots, \min(d - r, d - s)\}$, write

$$n_s^{ij} = \begin{cases} \sum_{m=0}^{j-i} \binom{d-i-m}{j-i-m} p_m^{(d,i)}(s), & i \leq j, \\ \sum_{m=0}^{i-j} \binom{i-m}{j} p_m^{(d,i)}(s), & i \geq j. \end{cases}$$

Similar to the proof of (12), we have

$${}^{rs}T_{ij} {}^{rs}T_{jl} = \sqrt{\frac{n_r^{ij} n_s^{ij} n_r^{jl} n_s^{jl}}{n_r^{il} n_s^{il}}} {}^{rs}T_{il}. \quad (17)$$

Let ϕ be the linear mapping from ${}^{rs}\mathcal{T}$ to $\text{Mat}_{d_{r,s}+1}(\mathbb{C})$ satisfying $\phi(L({}^{rs}T_{ij})) = \sqrt{n_r^{ij} n_s^{ij}} E_{i,j}$. By (17), we obtain the following result.

Theorem 5.4 *Let \mathcal{T} be the Terwilliger algebra of the Johnson scheme $J(2d, d)$. Then*

$$\mathcal{T} \simeq \bigoplus_{s=0}^{\lfloor d/2 \rfloor} \bigoplus_{r=0}^s \text{Mat}_{d_{r,s}+1}(\mathbb{C}).$$

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